OXFORD
MATHEMATICS

Algebraic Geometry and Arithmetic Curves

Qing Liu

OXFORD GRADUATE TEXTS IN MATHEMATICS | 6

Oxford Graduate Texts in Mathematics

OXFORD GRADUATE TEXTS IN MATHEMATICS

---

1. Keith Hannabuss: *An Introduction to Quantum Theory*
2. Reinhold Meise and Dietmar Vogt: *Introduction to Functional Analysis*
3. James G. Oxley: *Matroid Theory*
4. N.J. Hitchin, G.B. Segal, and R.S. Ward: *Integrable Systems: Twistors, Loop Groups, and Riemann Surfaces*
5. Wulf Rossmann: *Lie groups: An Introduction Through Linear Groups*
6. Q. Liu: *Algebraic Geometry and Arithmetic Curves*
7. Martin R. Bridson and Simon M, Salamon (eds): *Invitations to Geometry and Topology*
8. Shmuel Kantorovitz: *Introduction to Modern Analysis*
9. Terry Lawson: *Topology: A Geometric Approach*
10. Meinolf Geck: *An Introduction to Algebraic Geometry and Algebraic Groups*

Algebraic Geometry and Arithmetic Curves

Qing Liu
Professor
CNRS Laboratoire de Théorie des Nombres et d'Algorithmique Arithmétique
Université Bordeaux 1

Translated by

Reinie Erné
Institut de Recherche Mathématique de Rennes
Université Rennes 1

OXFORD
UNIVERSITY PRESS

OXFORD
UNIVERSITY PRESS

Great Clarendon Street, Oxford OX2 6DP

Oxford University Press is a department of the University of Oxford.
It furthers the University's objective of excellence in research, scholarship,
and education by publishing worldwide in

Oxford New York

Auckland Cape Town Dar es Salaam Hong Kong Karachi
Kuala Lumpur Madrid Melbourne Mexico City Nairobi
New Delhi Shanghai Taipei Toronto

With offices in

Argentina Austria Brazil Chile Czech Republic France Greece
Guatemala Hungary Italy Japan Poland Portugal Singapore
South Korea Switzerland Thailand Turkey Ukraine Vietnam

Oxford is a registered trade mark of Oxford University Press
in the UK and in certain other countries

Published in the United States
by Oxford University Press Inc., New York

First published 2002
First published in paperback 2006

British Library Cataloguing in Publication Data
Data available

Library of Congress Cataloging in Publication Data
Data available

Typeset by Newgen Imaging Systems (P) Ltd, Chennai, India
Printed in Great Britain
on acid-free paper by
Biddles Ltd, King's Lynn

ISBN 0–19–850284–2 978–0–19–850284–5
ISBN 0–19–920249–4 (Pbk.) 978–0–19–920249–2 (Pbk.)

1 3 5 7 9 10 8 6 4 2

To my mother

This page intentionally left blank

Preface

This book begins with an introduction to algebraic geometry in the language of schemes. Then, the general theory is illustrated through the study of arithmetic surfaces and the reduction of algebraic curves. The origin of this work is notes distributed to the participants of a course on arithmetic surfaces for graduate students. The aim of the course was to describe the foundation of the geometry of arithmetic surfaces as presented in [56] and [90], and the theory of stable reduction [26]. In spite of the importance of recent developments in these subjects and of their growing implications in number theory, unfortunately there does not exist any book in the literature that treats these subjects in a systematic manner, and at a level that is accessible to a student or to a mathematician who is not a specialist in the field. The aim of this book is therefore to gather together these results, now classical and indispensable in arithmetic geometry, in order to make them more easily accessible to a larger audience.

The first part of the book presents general aspects of the theory of schemes. It can be useful to a student of algebraic geometry, even if a thorough examination of the subjects treated in the second part is not required. Let us briefly present the contents of the first seven chapters that make up this first part. I believe that we cannot separate the learning of algebraic geometry from the study of commutative algebra. That is the reason why the book starts with a chapter on the tensor product, flatness, and formal completion. These notions will frequently recur throughout the book. In the second chapter, we begin with Hilbert's Nullstellensatz, in order to give an intuitive basis for the theory of schemes. Next, schemes and morphisms of schemes, as well as other basic notions, are defined. In Chapter 3, we study the fibered product of schemes and the fundamental concept of base change. We examine the behavior of algebraic varieties with respect to base change, before going on to proper morphisms and to projective morphisms. Chapter 4 treats local properties of schemes and of morphisms such as normality and smoothness. We conclude with an elementary proof of Zariski's Main Theorem. The global aspect of schemes is approached through the theory of coherent sheaves in Chapter 5. After studying coherent sheaves on projective schemes, we define the Čech cohomology of sheaves, and we look at some fundamental theorems such as Serre's finiteness theorem, the theorem of formal functions, and as an application, Zariski's connectedness principle. Chapter 6 studies particular coherent sheaves: the sheaf of differentials, and, in certain favorable cases (local complete intersections), the relative dualizing sheaf. At the end of that chapter, we present Grothendieck's duality theory. Chapter 7 starts with a rather general study of divisors, which is then restricted to the case of projective curves over a field. The theorem of Riemann–Roch, as well as Hurwitz's theorem, are proven with the help of duality theory. The chapter concludes with a detailed study of the Picard group of a not necessarily reduced projective curve over an algebraically closed field. The necessity of studying singular curves arises, among other things, from the fact that an arithmetic (hence regular) surface in general has fibers that are singular. These seven chapters can be used for a basic course on algebraic geometry.

The second part of the book is made up of three chapters. Chapter 8 begins with the study of blowing-ups. An intermediate section digresses towards commutative algebra by giving, often without proof, some principal results concerning Cohen–Macaulay, Nagata, and excellent rings. Next, we present the general aspects of fibered surfaces over a Dedekind ring and the theory of desingularization of surfaces. Chapter 9 studies intersection theory on an arithmetic surface, and its applications. In particular, we show the adjunction formula, the factorization theorem, Castelnuovo's criterion, and the existence of the minimal regular model. The last chapter treats the reduction theory of algebraic curves. After discussing general properties that essentially follow from the study of arithmetic surfaces, we treat the different types of reduction of elliptic curves in detail. The end of the chapter is devoted to stable curves and stable reduction. We describe the proof of the stable reduction theorem of Deligne–Mumford by Artin–Winters, and we give some concrete examples of computations of the stable reduction.

From the outset, the book was written with arithmetic geometry in mind. In particular, we almost never suppose that the base field is algebraically closed, nor of characteristic zero, nor even perfect. Likewise, for the arithmetic surfaces, in general we do not impose any hypothesis on the base (Dedekind) rings. In fact, it does not demand much effort to work in general conditions, and does not affect the presentation in an unreasonable way. The advantage is that it lets us acquire good reflexes right from the beginning.

As far as possible, the treatment is self-contained. The prerequisites for reading this book are therefore rather few. A good undergraduate student, and in any case a graduate student, possesses, in principle, the background necessary to begin reading the book. In addressing beginners, I have found it necessary to render concepts explicit with examples, and above all exercises. In this spirit, all sections end with a list of exercises. Some are simple applications of already proven results, others are statements of results which did not fit in the main text. All are sufficiently detailed to be solved with a minimum of effort. This book should therefore allow the reader to approach more specialized works such as [25] and [15] with more ease.

Acknowledgements

It is my great pleasure to thank Michel Matignon and Martin Taylor, who encouraged me to write up my lecture notes. Reinie Erné combined her linguistic and mathematical talents to translate this book from French to English. I thank her for her patience and generous help. I thank Philippe Cassou-Noguès, Reinie Erné, Arnaud Lacoume, Thierry Sageaux, Alain Thiéry, and especially Dino Lorenzini and Sylvain Maugeais for their careful reading of the manuscript. It is due to their vigilance that many errors were found and corrected. My thanks also go to Jean Fresnel, Dino Lorenzini, and Michel Matignon for mathematical discussions during the preparation of the book. I thank the Laboratoire de Mathématiques Pures de Bordeaux for providing me with such an agreeable environment for the greatest part of the writing of this book.

I cannot thank my friends and family enough for their constant encouragement and their understanding. I apologize for not being able to name them individually. Finally, special thanks to Isabelle, who supported me and who put up with me during the long period of writing. Without her sacrifices and the encouragement that she gave me in moments of doubt, this book would probably be far from being finished today.

Numbering style

The book is organized by chapter/section/subsection. Each section ends with a series of exercises. The statements and exercises are numbered within each section. References to results and definitions consist of the chapter number followed by the section number and the reference number within the section. The first one is omitted when the reference is to a result within the same chapter. Thus a reference to Proposition 2.7; 3.2.7; means, respectively, Section 2, Proposition 2.7 of the same chapter; and Chapter 3, Section 2, Proposition 2.7. On the contrary, we always refer to sections and subsections with the chapter number followed by the section number, and followed by the subsection number for subsections: e.g., Section 3.2 and Subsection 3.2.4.

Errata

Future errata will be listed at
http://www.math.u-bordeaux.fr/~liu/Book/errata.html

Q.L.
Bordeaux
June 2001

Preface to the paper-back edition

I am very much indebted to many people who have contributed comments and corrections since this book was first published in 2002. My hearty thanks to Robert Ash, Michael Brunnbauer, Oliver Dodane, Rémy Eupherte, Xander Faber, Anton Geraschenko, Yves Laszlo, Yogesh More, and especially to Lars Halvard Halle, Carlos Ivorra, Dino Lorenzini and René Schmidt.

The list of all changes made from the first edition is found on my web page
http://www.math.u-bordeaux.fr/ liu/Book/errata.html
This web page will also include the list of errata for the present edition.

Q.L.
Bordeaux
March 2006

Contents

This page intentionally left blank

1

Some topics in commutative algebra

Unless otherwise specified, all rings in this book will be supposed commutative and with unit.

In this chapter, we introduce some indispensable basic notions of commutative algebra such as the tensor product, localization, and flatness. Other, more elaborate notions will be dealt with later, as they are needed. We assume that the reader is familiar with linear algebra over a commutative ring, and with Noetherian rings and modules.

1.1 Tensor products

In the theory of schemes, the fibered product plays an important role (in particular the technique of base change). The corresponding notion in commutative algebra is the tensor product of modules over a ring.

1.1.1 Tensor product of modules

Definition 1.1. Let A be a commutative ring with unit. Let M, N be two A-modules. The *tensor product of M and N over A* is defined to be an A-module H, together with a bilinear map $\phi : M \times N \to H$ satisfying the following universal property:

> For every A-module L and every bilinear map $f : M \times N \to L$, there exists a unique homomorphism of A-modules $\tilde{f} : H \to L$ making the following diagram commutative:

$$\begin{array}{ccc} M \times N & \xrightarrow{f} & L \\ {\scriptstyle\phi}\downarrow & \nearrow_{\tilde{f}} & \\ H & & \end{array}$$

Proposition 1.2. *Let A be a ring, and let M, N be A-modules. The tensor product (H, ϕ) exists, and is unique up to isomorphism.*

Proof As the solution of a universal problem, the uniqueness is automatic, and its proof is standard. We give it here as an example. Let (H, ϕ) and (H', ϕ') be two solutions. By the universal property, ϕ and ϕ' factor respectively as $\phi = \tilde{\phi} \circ \phi'$ and $\phi' = \tilde{\phi}' \circ \phi$. It follows that $\phi = (\tilde{\phi} \circ \tilde{\phi}') \circ \phi$. As $\phi = \mathrm{Id} \circ \phi$, it follows from the uniqueness of the decomposition of ϕ that $(\tilde{\phi} \circ \tilde{\phi}') = \mathrm{Id}$. Thus we see that $\tilde{\phi} : H \to H'$ is an isomorphism.

Let us now show existence. Consider the free A-module $A^{(M \times N)}$ with basis $M \times N$. Let $\{e_{x,y}\}_{(x,y) \in M \times N}$ denote its canonical basis. Let L be the submodule of $A^{(M \times N)}$ generated by the elements having one of the following forms:

$$\begin{cases} e_{x_1+x_2,y} - e_{x_1,y} - e_{x_2,y} \\ e_{x,y_1+y_2} - e_{x,y_1} - e_{x,y_2} \\ e_{ax,y} - e_{x,ay}, \quad ae_{x,y} - e_{ax,y}, \quad a \in A. \end{cases}$$

Let $H = A^{(M \times N)}/L$, and $\phi : M \times N \to H$ be the map defined by $\phi(x, y) =$ the image of $e_{x,y}$ in H. One immediately verifies that the pair (H, ϕ) verifies the universal property mentioned above. □

Notation. We denote the tensor product of M and N by $(M \otimes_A N, \phi)$. In general, the map ϕ is omitted in the notation. For any $(x, y) \in M \times N$, we let $x \otimes y$ denote its image by ϕ. By the bilinearity of ϕ, we have $a(x \otimes y) = (ax) \otimes y = x \otimes (ay)$ for every $a \in A$.

Remark 1.3. By construction, $M \otimes_A N$ is generated as an A-module by its elements of the form $x \otimes y$. Thus every element of $M \otimes_A N$ can be written (though not in a unique manner) as a finite sum $\sum_i x_i \otimes y_i$, with $x_i \in M$ and $y_i \in N$. In general, an element of $M \otimes_A N$ cannot be written $x \otimes y$.

Example 1.4. Let $A = \mathbb{Z}$, $M = A/2A$, and $N = A/3A$. Then $M \otimes_A N = 0$. In fact, for every $(x, y) \in M \times N$, we have $x \otimes y = 3(x \otimes y) - 2(x \otimes y) = x \otimes (3y) - (2x) \otimes y = 0$.

Proposition 1.5. *Let A be a ring, and let M, N, M_i be A-modules. We have the following canonical isomorphisms of A-modules:*

(a) $M \otimes_A A \simeq M$;

(b) *(commutativity)* $M \otimes_A N \simeq N \otimes_A M$;

(c) *(associativity)* $(L \otimes_A M) \otimes_A N \simeq L \otimes_A (M \otimes_A N)$;

(d) *(distributivity)* $(\oplus_{i \in I} M_i) \otimes_A N \simeq \oplus_{i \in I}(M_i \otimes_A N)$.

Proof Everything follows from the universal property. Let us, for example, show (a) and (d).

(a) Let $\phi : M \times A \to M$ be the bilinear map defined by $(x, a) \mapsto ax$. For any bilinear map $f : M \times A \to L$, set $\tilde{f} : M \to L, x \mapsto f(x, 1)$. Then $f = \tilde{f} \circ \phi$, and $\tilde{f}$ is the unique linear map $M \to L$ having this property. Hence (M, ϕ) is the tensor product of M and A.

(d) Let $\phi : (\oplus_{i\in I} M_i) \times N \to \oplus_{i\in I}(M_i \otimes_A N)$ be the map defined by

$$\phi : \Big(\sum_i x_i, y\Big) \mapsto \sum_i (x_i \otimes y).$$

Let $f : (\oplus_{i\in I} M_i) \times N \to L$ be a bilinear map. For every $i \in I$, f induces a bilinear map $f_i : M_i \times N \to L$ which factors through $\widetilde{f_i} : M_i \otimes_A N \to L$. One verifies that f factors uniquely as $f = \tilde{f} \circ \psi$, where $\psi : (\oplus_{i\in I} M_i) \times N \to (\oplus_{i\in I} M_i) \otimes N$ is the canonical map and $\tilde{f} = \oplus_i \widetilde{f_i}$. Hence $\oplus_{i\in I}(M_i \otimes_A N)$ is the tensor product of $(\oplus_{i\in I} M_i)$ with N. □

Corollary 1.6. *Let M be a free A-module with basis $\{e_i\}_{i\in I}$. Then every element of $M \otimes_A N$ can be written uniquely as a finite sum $\sum_i e_i \otimes y_i$, with $y_i \in N$. In particular, if A is a field and $\{e_i\}_{i\in I}$ (resp. $\{d_j\}_{j\in J}$) is a basis of M (resp. of N), then $\{e_i \otimes d_j\}_{(i,j)\in I\times J}$ is a basis of $M \otimes_A N$.*

Remark 1.7. The associativity of the tensor product allows us to define the tensor product $M_1 \otimes_A \cdots \otimes_A M_n$ of a finite number of A-modules. This tensor product has a universal property analogous to that of the tensor product of two modules, with the bilinear maps replaced by multilinear ones.

Definition 1.8. Let $u : M \to M'$, $v : N \to N'$ be linear maps of A-modules. By the universal property of the tensor product, there exists a unique A-linear map $u \otimes v : M \otimes_A N \to M' \otimes_A N'$ such that $(u \otimes v)(x \otimes y) = u(x) \otimes v(y)$. In fact, the map $g : M \times N \to M' \otimes_A N'$ defined by $g(x, y) = u(x) \otimes v(y)$ is clearly bilinear, and hence factors uniquely as $(u \otimes v) \circ \phi$, where ϕ is the canonical map $M \times N \to M \otimes N$. The map $u \otimes v$ is called the *tensor product of u and v*. The notation is justified by Exercise 1.2.

Let $\rho : A \to B$ be a ring homomorphism, and N a B-module. Then ρ induces, in a natural way, the structure of an A-module on N: for any $a \in A$ and $y \in N$, we set $a \cdot y = \rho(a)y$. We denote this A-module by $\rho_* N$, or simply by N.

Definition 1.9. Let M be an A-module. We can endow $M \otimes_A N$ with the structure of a B-module as follows. Let $b \in B$. Let $t_b : N \to N$ denote the multiplication by b, and for any $z \in M \otimes_A N$, set $b \cdot z := (\mathrm{Id}_M \otimes t_b)(z)$. One easily verifies that this defines the structure of a B-module. We denote the B-module $M \otimes_A B$ by $\rho^* M$. This is called the *extension of scalars of M by B*. By construction, we have $b(x \otimes y) = x \otimes (by)$ for every $b \in B$, $x \in M$, and $y \in N$.

Proposition 1.10. *Let $\rho : A \to B$ be a ring homomorphism, M an A-module, and let N, P be B-modules. Then there exists a canonical isomorphism of B-modules*

$$M \otimes_A (N \otimes_B P) \simeq (M \otimes_A N) \otimes_B P.$$

Proof Let us show that there exist A-linear maps

$$f : M \otimes_A (N \otimes_B P) \to (M \otimes_A N) \otimes_B P, \quad g : (M \otimes_A N) \otimes_B P \to M \otimes_A (N \otimes_B P)$$

such that for every $x \in M$, $y \in N$, and $z \in P$, we have $f(x \otimes (y \otimes z)) = (x \otimes y) \otimes z$ and $g((x \otimes y) \otimes z) = x \otimes (y \otimes z)$. This will imply that f is an isomorphism, with inverse g. The B-linearity of f follows from this identity.