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Algebraic Geometry and Arithmetic Curves

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To my mother

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Preface

This book begins with an introduction to algebraic geometry in the language of schemes. Then, the general theory is illustrated through the study of arithmetic surfaces and the reduction of algebraic curves. The origin of this work is notes distributed to the participants of a course on arithmetic surfaces for graduate students. The aim of the course was to describe the foundation of the geometry of arithmetic surfaces as presented in [56] and [90], and the theory of stable reduction [26]. In spite of the importance of recent developments in these subjects and of their growing implications in number theory, unfortunately there does not exist any book in the literature that treats these subjects in a systematic manner, and at a level that is accessible to a student or to a mathematician who is not a specialist in the field. The aim of this book is therefore to gather together these results, now classical and indispensable in arithmetic geometry, in order to make them more easily accessible to a larger audience.

The first part of the book presents general aspects of the theory of schemes. It can be useful to a student of algebraic geometry, even if a thorough examination of the subjects treated in the second part is not required. Let us briefly present the contents of the first seven chapters that make up this first part. I believe that we cannot separate the learning of algebraic geometry from the study of commutative algebra. That is the reason why the book starts with a chapter on the tensor product, flatness, and formal completion. These notions will frequently recur throughout the book. In the second chapter, we begin with Hilbert's Nullstellensatz, in order to give an intuitive basis for the theory of schemes. Next, schemes and morphisms of schemes, as well as other basic notions, are defined. In Chapter 3, we study the fibered product of schemes and the fundamental concept of base change. We examine the behavior of algebraic varieties with respect to base change, before going on to proper morphisms and to projective morphisms. Chapter 4 treats local properties of schemes and of morphisms such as normality and smoothness. We conclude with an elementary proof of Zariski's Main Theorem. The global aspect of schemes is approached through the theory of coherent sheaves in Chapter 5. After studying coherent sheaves on projective schemes, we define the Čech cohomology of sheaves, and we look at some fundamental theorems such as Serre's finiteness theorem, the theorem

of formal functions, and as an application, Zariski's connectedness principle. Chapter 6 studies particular coherent sheaves: the sheaf of differentials, and, in certain favorable cases (local complete intersections), the relative dualizing sheaf. At the end of that chapter, we present Grothendieck's duality theory. Chapter 7 starts with a rather general study of divisors, which is then restricted to the case of projective curves over a field. The theorem of Riemann–Roch, as well as Hurwitz's theorem, are proven with the help of duality theory. The chapter concludes with a detailed study of the Picard group of a not necessarily reduced projective curve over an algebraically closed field. The necessity of studying singular curves arises, among other things, from the fact that an arithmetic (hence regular) surface in general has fibers that are singular. These seven chapters can be used for a basic course on algebraic geometry.

The second part of the book is made up of three chapters. Chapter 8 begins with the study of blowing-ups. An intermediate section digresses towards commutative algebra by giving, often without proof, some principal results concerning Cohen–Macaulay, Nagata, and excellent rings. Next, we present the general aspects of fibered surfaces over a Dedekind ring and the theory of desingularization of surfaces. Chapter 9 studies intersection theory on an arithmetic surface, and its applications. In particular, we show the adjunction formula, the factorization theorem, Castelnuovo's criterion, and the existence of the minimal regular model. The last chapter treats the reduction theory of algebraic curves. After discussing general properties that essentially follow from the study of arithmetic surfaces, we treat the different types of reduction of elliptic curves in detail. The end of the chapter is devoted to stable curves and stable reduction. We describe the proof of the stable reduction theorem of Deligne–Mumford by Artin–Winters, and we give some concrete examples of computations of the stable reduction.

From the outset, the book was written with arithmetic geometry in mind. In particular, we almost never suppose that the base field is algebraically closed, nor of characteristic zero, nor even perfect. Likewise, for the arithmetic surfaces, in general we do not impose any hypothesis on the base (Dedekind) rings. In fact, it does not demand much effort to work in general conditions, and does not affect the presentation in an unreasonable way. The advantage is that it lets us acquire good reflexes right from the beginning.

As far as possible, the treatment is self-contained. The prerequisites for reading this book are therefore rather few. A good undergraduate student, and in any case a graduate student, possesses, in principle, the background necessary to begin reading the book. In addressing beginners, I have found it necessary to render concepts explicit with examples, and above all exercises. In this spirit, all sections end with a list of exercises. Some are simple applications of already proven results, others are statements of results which did not fit in the main text. All are sufficiently detailed to be solved with a minimum of effort. This book should therefore allow the reader to approach more specialized works such as [25] and [15] with more ease.

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Numbering style

The book is organized by chapter/section/subsection. Each section ends with a series of exercises. The statements and exercises are numbered within each section. References to results and definitions consist of the chapter number followed by the section number and the reference number within the section. The first one is omitted when the reference is to a result within the same chapter. Thus a reference to Proposition 2.7; 3.2.7; means, respectively, Section 2, Proposition 2.7 of the same chapter; and Chapter 3, Section 2, Proposition 2.7. On the contrary, we always refer to sections and subsections with the chapter number followed by the section number, and followed by the subsection number for subsections: e.g., Section 3.2 and Subsection 3.2.4.

Errata

Future errata will be listed at
<http://www.math.u-bordeaux.fr/~liu/Book/errata.html>

Q.L.
Bordeaux
June 2001

Preface to the paper-back edition

I am very much indebted to many people who have contributed comments and corrections since this book was first published in 2002. My hearty thanks to Robert Ash, Michael Brunnbauer, Oliver Dodane, Rémy Eupherte, Xander Faber, Anton Geraschenko, Yves Laszlo, Yogesh More, and especially to Lars Halvard Halle, Carlos Ivorra, Dino Lorenzini and René Schmidt.

The list of all changes made from the first edition is found on my web page <http://www.math.u-bordeaux.fr/liu/Book/errata.html>

This web page will also include the list of errata for the present edition.

Q.L.

Bordeaux

March 2006

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1

Some topics in commutative algebra

Unless otherwise specified, all rings in this book will be supposed commutative and with unit.

In this chapter, we introduce some indispensable basic notions of commutative algebra such as the tensor product, localization, and flatness. Other, more elaborate notions will be dealt with later, as they are needed. We assume that the reader is familiar with linear algebra over a commutative ring, and with Noetherian rings and modules.

1.1 Tensor products

In the theory of schemes, the fibered product plays an important role (in particular the technique of base change). The corresponding notion in commutative algebra is the tensor product of modules over a ring.

1.1.1 Tensor product of modules

Definition 1.1. Let A be a commutative ring with unit. Let M, N be two A -modules. The *tensor product of M and N over A* is defined to be an A -module H , together with a bilinear map $\phi : M \times N \rightarrow H$ satisfying the following universal property:

For every A -module L and every bilinear map $f : M \times N \rightarrow L$, there exists a unique homomorphism of A -modules $\tilde{f} : H \rightarrow L$ making the following diagram commutative:

$$\begin{array}{ccc} M \times N & \xrightarrow{f} & L \\ \phi \downarrow & \nearrow \tilde{f} & \\ H & & \end{array}$$

Proposition 1.2. *Let A be a ring, and let M, N be A -modules. The tensor product (H, ϕ) exists, and is unique up to isomorphism.*

Proof As the solution of a universal problem, the uniqueness is automatic, and its proof is standard. We give it here as an example. Let (H, ϕ) and (H', ϕ') be two solutions. By the universal property, ϕ and ϕ' factor respectively as $\phi = \tilde{\phi} \circ \phi'$ and $\phi' = \tilde{\phi}' \circ \phi$. It follows that $\phi = (\tilde{\phi} \circ \tilde{\phi}') \circ \phi$. As $\phi = \text{Id} \circ \phi$, it follows from the uniqueness of the decomposition of ϕ that $(\tilde{\phi} \circ \tilde{\phi}') = \text{Id}$. Thus we see that $\tilde{\phi} : H \rightarrow H'$ is an isomorphism.

Let us now show existence. Consider the free A -module $A^{(M \times N)}$ with basis $M \times N$. Let $\{e_{x,y}\}_{(x,y) \in M \times N}$ denote its canonical basis. Let L be the submodule of $A^{(M \times N)}$ generated by the elements having one of the following forms:

$$\begin{cases} e_{x_1+x_2,y} - e_{x_1,y} - e_{x_2,y} \\ e_{x,y_1+y_2} - e_{x,y_1} - e_{x,y_2} \\ e_{ax,y} - e_{x,ay}, \quad ae_{x,y} - e_{ax,y}, \quad a \in A. \end{cases}$$

Let $H = A^{(M \times N)}/L$, and $\phi : M \times N \rightarrow H$ be the map defined by $\phi(x, y) =$ the image of $e_{x,y}$ in H . One immediately verifies that the pair (H, ϕ) verifies the universal property mentioned above. \square

Notation. We denote the tensor product of M and N by $(M \otimes_A N, \phi)$. In general, the map ϕ is omitted in the notation. For any $(x, y) \in M \times N$, we let $x \otimes y$ denote its image by ϕ . By the bilinearity of ϕ , we have $a(x \otimes y) = (ax) \otimes y = x \otimes (ay)$ for every $a \in A$.

Remark 1.3. By construction, $M \otimes_A N$ is generated as an A -module by its elements of the form $x \otimes y$. Thus every element of $M \otimes_A N$ can be written (though not in a unique manner) as a finite sum $\sum_i x_i \otimes y_i$, with $x_i \in M$ and $y_i \in N$. In general, an element of $M \otimes_A N$ cannot be written $x \otimes y$.

Example 1.4. Let $A = \mathbb{Z}$, $M = A/2A$, and $N = A/3A$. Then $M \otimes_A N = 0$. In fact, for every $(x, y) \in M \times N$, we have $x \otimes y = 3(x \otimes y) - 2(x \otimes y) = x \otimes (3y) - (2x) \otimes y = 0$.

Proposition 1.5. *Let A be a ring, and let M, N, M_i be A -modules. We have the following canonical isomorphisms of A -modules:*

- (a) $M \otimes_A A \simeq M$;
- (b) (commutativity) $M \otimes_A N \simeq N \otimes_A M$;
- (c) (associativity) $(L \otimes_A M) \otimes_A N \simeq L \otimes_A (M \otimes_A N)$;
- (d) (distributivity) $(\bigoplus_{i \in I} M_i) \otimes_A N \simeq \bigoplus_{i \in I} (M_i \otimes_A N)$.

Proof Everything follows from the universal property. Let us, for example, show (a) and (d).

(a) Let $\phi : M \times A \rightarrow M$ be the bilinear map defined by $(x, a) \mapsto ax$. For any bilinear map $f : M \times A \rightarrow L$, set $\tilde{f} : M \rightarrow L, x \mapsto f(x, 1)$. Then $f = \tilde{f} \circ \phi$, and \tilde{f} is the unique linear map $M \rightarrow L$ having this property. Hence (M, ϕ) is the tensor product of M and A .

(d) Let $\phi : (\oplus_{i \in I} M_i) \times N \rightarrow \oplus_{i \in I} (M_i \otimes_A N)$ be the map defined by

$$\phi : \left(\sum_i x_i, y \right) \mapsto \sum_i (x_i \otimes y).$$

Let $f : (\oplus_{i \in I} M_i) \times N \rightarrow L$ be a bilinear map. For every $i \in I$, f induces a bilinear map $f_i : M_i \times N \rightarrow L$ which factors through $\tilde{f}_i : M_i \otimes_A N \rightarrow L$. One verifies that f factors uniquely as $f = \tilde{f} \circ \psi$, where $\psi : (\oplus_{i \in I} M_i) \times N \rightarrow (\oplus_{i \in I} M_i) \otimes N$ is the canonical map and $\tilde{f} = \oplus_i \tilde{f}_i$. Hence $\oplus_{i \in I} (M_i \otimes_A N)$ is the tensor product of $(\oplus_{i \in I} M_i)$ with N . \square

Corollary 1.6. *Let M be a free A -module with basis $\{e_i\}_{i \in I}$. Then every element of $M \otimes_A N$ can be written uniquely as a finite sum $\sum_i e_i \otimes y_i$, with $y_i \in N$. In particular, if A is a field and $\{e_i\}_{i \in I}$ (resp. $\{d_j\}_{j \in J}$) is a basis of M (resp. of N), then $\{e_i \otimes d_j\}_{(i,j) \in I \times J}$ is a basis of $M \otimes_A N$.*

Remark 1.7. The associativity of the tensor product allows us to define the tensor product $M_1 \otimes_A \cdots \otimes_A M_n$ of a finite number of A -modules. This tensor product has a universal property analogous to that of the tensor product of two modules, with the bilinear maps replaced by multilinear ones.

Definition 1.8. Let $u : M \rightarrow M'$, $v : N \rightarrow N'$ be linear maps of A -modules. By the universal property of the tensor product, there exists a unique A -linear map $u \otimes v : M \otimes_A N \rightarrow M' \otimes_A N'$ such that $(u \otimes v)(x \otimes y) = u(x) \otimes v(y)$. In fact, the map $g : M \times N \rightarrow M' \otimes_A N'$ defined by $g(x, y) = u(x) \otimes v(y)$ is clearly bilinear, and hence factors uniquely as $(u \otimes v) \circ \phi$, where ϕ is the canonical map $M \times N \rightarrow M \otimes_A N$. The map $u \otimes v$ is called the *tensor product of u and v* . The notation is justified by Exercise 1.2.

Let $\rho : A \rightarrow B$ be a ring homomorphism, and N a B -module. Then ρ induces, in a natural way, the structure of an A -module on N : for any $a \in A$ and $y \in N$, we set $a \cdot y = \rho(a)y$. We denote this A -module by $\rho_* N$, or simply by N .

Definition 1.9. Let M be an A -module. We can endow $M \otimes_A N$ with the structure of a B -module as follows. Let $b \in B$. Let $t_b : N \rightarrow N$ denote the multiplication by b , and for any $z \in M \otimes_A N$, set $b \cdot z := (\text{Id}_M \otimes t_b)(z)$. One easily verifies that this defines the structure of a B -module. We denote the B -module $M \otimes_A N$ by $\rho^* M$. This is called the *extension of scalars of M by B* . By construction, we have $b(x \otimes y) = x \otimes (by)$ for every $b \in B$, $x \in M$, and $y \in N$.

Proposition 1.10. *Let $\rho : A \rightarrow B$ be a ring homomorphism, M an A -module, and let N, P be B -modules. Then there exists a canonical isomorphism of B -modules*

$$M \otimes_A (N \otimes_B P) \simeq (M \otimes_A N) \otimes_B P.$$

Proof Let us show that there exist A -linear maps

$$f : M \otimes_A (N \otimes_B P) \rightarrow (M \otimes_A N) \otimes_B P, \quad g : (M \otimes_A N) \otimes_B P \rightarrow M \otimes_A (N \otimes_B P)$$

such that for every $x \in M$, $y \in N$, and $z \in P$, we have $f(x \otimes (y \otimes z)) = (x \otimes y) \otimes z$ and $g((x \otimes y) \otimes z) = x \otimes (y \otimes z)$. This will imply that f is an isomorphism, with inverse g . The B -linearity of f follows from this identity.